Flow Dimension and Anomalous Diffusion of Aquifer Tests in Fracture Networks

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Characterization and modeling of aquifers is particularly challenging in fractured media, where flow is concentrated into channels that are poorly suited to traditional approaches to analysis. The generalized radial flow model is an alternative method for hydraulic test interpretation that infers an additional parameter, the flow dimension \( n \), to describe the flow geometry. This study was a Monte Carlo analysis of numerical models of aquifer tests in two-dimensional fractured media, with the objective of understanding the relationships between the parameters of a discrete fracture network (DFN) with lengths distributed as a power law, the flow dimension, and the regime of hydraulic diffusion (e.g., Fickian or non-Fickian). The diffusion regime of each realization was evaluated using \( \langle R_t^2 \rangle \sim t^k \), where \( t \) is time, \( \langle R_t^2 \rangle \) is the mean squared radius of hydraulic diffusion, and an apparent value of \( k = 1 \) indicates normal (Fickian) diffusion and \( k < 1 \) indicates anomalous (non-Fickian) diffusion. For the DFN model, the apparent flow dimension and exponent \( k \) depend on both the density and the power of the fracture length distribution, and thus also on the connectivity regime of the fracture network system. Depending on the connectivity regime, the apparent flow dimension stabilizes to less than the Euclidean dimension and the apparent value of \( k < 1 \) indicates that hydraulic diffusion is non-Fickian. These results suggest that the flow dimension and the exponent \( k \) may be useful for characterizing flow and transport in fractured media.

**ABBREVIATIONS:** DFN, discrete fracture network; GRF, generalized radial flow; mvG, multivariate Gaussian.

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An alternative understanding of fractured rock aquifers is the generalized radial flow (GRF) model, which fits a parameter known as the flow dimension, \( n \), to describe the change in the cross-sectional area of flow with radial distance from the pumped well (Barker, 1988). Where heterogeneities restrict flow into erratic channels, the flow dimension can be less than the spatial (Euclidean) dimension of the aquifer, a phenomenon that has been observed at many field sites (Acuna and Yortsos, 1995; Bangoy et al., 1992; de Dreuzy and Davy, 2007). Unlike the parameters inferred from traditional interpretations of aquifer tests, the relationships among the flow dimension, aquifer heterogeneity, and fracture connectivity have received comparatively little attention in the literature.

This study examined the connectivity, flow dimension, and time scaling of diffusion for a two-dimensional aquifer whose heterogeneity is represented by a DFN with a length distribution following a power law of the form

\[
n(l) = \beta l^{-a}
\]

where \( n(l) \) is the number of faults having a length in the range \( [l, l + dl] \), \( \beta \) is a density coefficient that controls the intensity of fracturing in the system, and \( a \) is an exponent, typically inferred from observations, that determines how rapidly the number of fractures declines with length, \( l \).

This length law introduces long-range spatial correlations that are consistent with the fractal nature of observed fractures. The emphasis of this study was on fracture connectivity regimes that produce the noninteger flow dimensions inferred from...
A variety of models have been developed to address flow and transport in fractured rock aquifers. Approaches include dual-porosity models (Warren and Root, 1963), the multiple continua approach (Pruess and Narasimhan, 1985), and the DFN method (Cacas et al., 1990). The first two models represent the fractured system as effective continua, but typical applications of these models do not describe the multiple length scales observed in naturally fractured systems. The third model, the DFN, represents fractures as discrete linear features whose connectivity and flow properties depend on the distribution and shape of the features. Regardless of the conceptualization of fractures in models, methods for characterizing and modeling fractured media have lagged (Committee on Fracture Characterization and Fluid Flow, 1996).

The interpretation of aquifer tests for the characterization of fractured rock aquifers requires an approach that addresses the complex geometry of flow. Barker (1988) introduced \( n \) to generalize the traditional interpretation model of radial flow to a pumping well during an aquifer test. In the resulting GRF model, \( n \) describes how the cross-sectional area of the flow changes with radial distance from the pumped well:

\[
A(r) = b^{3-n} \frac{2n^{n/2}}{\Gamma(n/2)} r^{n-1}
\]

where \( A \) is the area, \( r \) is the radial distance, \( b \) is the thickness of the aquifer, and \( \Gamma \) is the gamma function. The flow dimension can be inferred from a constant-rate aquifer test in a nonleaky, infinite-acting aquifer using the log–log diagnostic plot of Bourdet et al. (1983). For such tests, the late-time slope, \( \nu \), of the log–log plot of the derivative of drawdown is related to the apparent flow dimension by

\[
n^*(t) = 2 - 2\nu(t)
\]

where

\[
\nu(t) = \frac{d}{d(\log t)} \left[ \log \left( \frac{dh}{d\ln t} \right) \right]
\]

(Barker, 1988; Mishra et al., 1991). For a constant-rate aquifer test in a homogeneous, infinite, two-dimensional domain, \( n = 2 \) by definition, which requires the late-time slope of the drawdown derivative to be \( \nu(t) = 0 \). The apparent flow dimension is a function of time because heterogeneities can cause the apparent flow dimension to vary as the aquifer test progresses, and some types of heterogeneity do not stabilize to a constant flow dimension (Walker and Roberts, 2003). Where strong heterogeneities concentrate flow into erratic channels, the flow dimension can be less than the Euclidean dimension of the aquifer. Figure 1 presents data from a constant-rate aquifer test in a fractured dolomite aquifer, shown as a log–log diagnostic plot of the drawdown and its semilog derivative vs. time (Bourdet et al., 1983). The test data is superposed with a traditional interpretation model of radial flow in two dimensions (Theis, 1935). Although the model nearly fits the drawdown data, the derivative shows that the slope of the model is a poor match to the data throughout this test. The poor fit of the model to the data suggests an unreliable conceptual model and increased uncertainties for the estimated parameters. The drawdown derivative in Fig. 1 has an average slope of \( \nu \sim 0.15 \), so that \( n \sim 1.7 \) on average, even though the aquifer arguably is two dimensional. In the absence of boundary effects, flow dimensions less than the Euclidean dimension are typically attributed to some unspecified form of heterogeneity.

To date, there have been few studies of the relationship between the flow dimension and models of aquifer heterogeneity. Barker (1988) suggested that observations of noninteger flow dimensions could be caused by a fractal network of conduits, and this was later numerically verified by Polek (1990) using percolation clusters and Sierpinski gaskets, objects known to have fractal geometries. Acuna and Yortsos (1995) examined the radial scaling of conductive features in a lattice using

\[
M(R) \sim R^{D_M}
\]

where \( M(R) \) is the mass (number) of conductive features inside a circle of radius \( R \) and \( D \) is the mass fractal dimension of the lattice (Mandelbrot, 1983). Acuna and Yortsos (1995) analyzed aquifer tests in lattices composed of fractal geometric shapes and showed that the flow dimension was a function of \( D \) and a connectivity factor that accounted for anomalously slow diffusion within a fractal lattice. They also found that \( n = D \) for a perfectly connected lattice. Doughty and Karasaki (2002) examined the flow dimension and transport in a two-dimensional domain populated with stochastic realizations of fractal geometric shapes. Doe (1991) argued that the observed flow dimensions could be explained by flow geometry, heterogeneous hydraulic conductivity, or combinations of the two. Walker and Roberts (2003) determined the flow dimension for several deterministic models of heterogeneity. Where all of the preceding studies tended to look at the flow dimension of idealized geometric objects, Walker et al. (2006b) examined stochastic models of heterogeneity. They

![Fig. 1. An aquifer test in a fractured dolomite aquifer for Hopkins Park Well no. 2, Kankakee County, Illinois, from Illinois State Water Survey archives. Circles denote the drawdown, \( s \), vs. time; triangles denote the derivative, \( s' \). The Theis model for radial flow is the solid line, and its derivative is the dashed line.](image-url)
found that log transmissivity fields distributed as a multivariate Gaussian process of moderate variance and as a fractional Brownian process do not consistently produce flow dimensions less than the Euclidean dimension of the aquifer. Walker et al. (2006b) also found that, for a site percolation network slightly above the percolation threshold, the apparent flow dimension oscillates around 1.6 and then tends to 2.0 when the scale of the aquifer test exceeds the correlation length of the percolation cluster, consistent with percolation model theory (Stauffer and Aharony, 1994). Walker et al. (2006b) speculated that a DFN model with lengths distributed as a power law might result in noninteger flow dimensions persisting for the duration of an aquifer test. Although there is some suggestion that the intensity of fracturing is positively correlated with the flow dimension (T. Doe, personal communication, 2006), a literature search failed to reveal studies that directly relate the flow dimension to the length distributions of fractures.

Aquifer tests are mathematically equivalent to the radial diffusion of heat or contaminants in heterogeneous media, and as such have been extensively studied. Havlin and Ben-Avraham (1987) noted that the mean squared radius of displacement, \( \langle R^2 \rangle \), of particles released from a source could be used to characterize radial diffusion:

\[
\langle R^2 \rangle \sim t^k
\]  

[6]

In a homogeneous medium, we would expect normal (Fickian) diffusion with a characteristic exponent of \( k = 1 \). For disordered media, Havlin and Ben-Avraham noted that anomalous (slow) diffusion has often been reported, with \( k < 1 \). If diffusion is modeled as a random walk in disordered media, the fractal dimension of the walk, \( D_w \), is \( >2 \) and diffusion is anomalously slow, with \( \langle R^2 \rangle \) scaling with time as \( k = 2/D_w \) (Sahimi, 1995, 1996) and Saadatfar and Sahimi (2002) examined radial diffusion in various types of fractal media including permeabilities with long-range correlations, and identified additional diffusive behaviors: super-diffusive with \( k > 1 \) and oscillating with \( k = f(t) \). Saadatfar and Sahimi (2002) did not, however, relate these diffusion behaviors to aquifer tests or the flow dimension. The connectivity factor of the model of Acuna and Yortsos (1995) allows for anomalous diffusion in addition to the geometric effects of hydraulic diffusion on a fractal lattice. Acuna and Yortsos (1995) determined the flow dimension, connectivity factor \( D_w \), and the mass fractal dimension for several fractal geometries by combining two different numerical approaches. The first approach consisted in estimating the slopes of both the drawdown and the drawdown derivative from simulated transient single-phase flow aquifer tests using the analytical solution of O’Shaughnessy and Proccacia (1985) for the anomalous diffusion equation. The second approach consisted in applying a random walk procedure to solve the diffusion problem. Combining the slopes obtained in the first approach with the scaling exponent of diffusion \( k \) obtained from the second approach using Eq. [6], they determined the corresponding parameters to characterize fracture geometry and pressure transient diffusion in fractal media as \( n = 2D/D_w \). They also verified the results by estimating the fractal dimension of the lattice. Le Borgne et al. (2004) successfully applied the model of Acuna and Yortsos (1995) to field data, but the relationship of model parameters to observed fracture characteristics remains largely unexplored.

Fractal models are natural candidates for describing fractured rocks because observed fracture lengths are widely reported to have broad distributions (such as a power law) and have no characteristic length scale (Turcotte, 1986; Bour and Davy, 1997; Bonnet et al., 2001; Sahimi, 1995). Bonnet et al. (2001) noted that \( D \sim 1.7 \) is typically reported for two-dimensional networks of fractures with a power-law exponent of \( -2 \). Bour and Davy (1997) applied concepts from percolation theory to power-law networks of fractures in a two-dimensional domain and identified three regimes of connectivity depending on the exponent in the power–length-law distribution given by Eq. [1]. For \( a > 3 \), the connectivity is ruled by fractures smaller than the system size; for \( a < 1 \) the connectivity of the fracture network is ruled by the largest fractures in the system; and for \( 1 < a < 3 \), the connectivity is a function of \( a \) and the proportion of large vs. small fractures. They also showed that under a connectivity characterized by \( 1 < a < 3 \) and a constant fracture density, the percolation parameter is no longer scale independent and that the connectivity strongly depends on a characteristic length \( L_w \). Thus, at scales smaller than \( L_w \), the fracture network is globally below the percolation threshold, while at scales above \( L_w \) the fracture network possesses trivial scaling properties similar to those observed by Walker et al. (2006a) for a site percolation network. In addition, Bour and Davy (1997) found that for an exponent \( a > 3 \), the connected cluster at the percolation threshold had \( D = 1.9 \) in agreement with percolation theory, and that this was independent of \( a \). For \( 2 < a < 3 \), Bour and Davy (1999) showed that under certain conditions, \( D = a - 1 \), in agreement with King (1983) and Turcotte (1986). Subsequent studies of permeability for each of these regimes (de Dreuzy et al., 2001) developed relationships for the scale dependency of the effective permeability and the validity of the effective medium approximation but did not examine the flow dimension.

**Approach**

This study used a Monte Carlo analysis of numerical models of aquifer tests to analyze the flow dimension and hydraulic diffusion, using an approach similar to that of Walker et al. (2006b). The steps in the analysis were: (i) create a transmissivity field \( T(x) \) using a DFN model with a power-law distribution of fracture lengths; (ii) simulate a constant-rate aquifer test using a finite-difference model of transient groundwater flow; (iii) estimate the apparent flow dimension \( n(t) \), for the centrally located pumping well; (iv) estimate the mean square radius of displacement of a diffusive particle \( \langle R^2 \rangle \) using a geometrical approach and the apparent diffusion coefficient \( k^2(t) \) at each time step of the aquifer test; and (v) estimate the mass fractal dimension \( D \) of the fracture network as a function of \( \langle R^2 \rangle \).

The Monte Carlo sequence is repeated for many realizations to infer the behavior of the flow dimension, the scaling of \( \langle R^2 \rangle \) with time, and the mass fractal dimension. Each realization is computationally independent, making the analysis well suited to distributed computing environments. For this project, the Monte Carlo simulations were performed using computing resources from the TeraGrid project (www.teragrid.org/, verified 4 Nov. 2008).

The aquifer test was simulated with MODFLOW-2000 (Harbaugh et al., 2000), the USGS finite-difference model for transient groundwater flow. The models of aquifer heterogeneity
were created by adapting algorithms from GSLIB (Deutsch and Journel, 1998).

Model of Aquifer Heterogeneity (Step 1)

The fractured rock aquifer was represented by a DFN model of linear features with homogeneous conductivity whose lengths follow a power law, set within a matrix of low permeability. In this study, we used the Boolean approach, where a stochastic point process generates the centroids of linear features or fractures, which are attached or “marked” to additional random processes defining the type, shape, size, and orientation of the linear features. While the centroids of the linear features representing fractures are randomly located according to a Poisson distribution, the required statistics for the fracture lengths and fracture orientations can be inferred in principle from sample distributions of field data. Then, the discrete features representing the fractures of high permeability are embedded in a continuous background matrix of low permeability. Since we used MODFLOW, it was necessary to discretize the overall domain into cells, which represent either a fracture or the background matrix. Hydrologic properties such as transmissivity, storativity, etc., were assigned to the cells representing either the fractures or matrix. An extended “virtual” transmissivity field surrounding the model domain was used to reduce edge effects due to long fractures with their centroids located outside of the model domain. Fracture lengths were distributed according to the power-law model (Eq. [1]) using a procedure similar to Bour and Davy (1997), with fracture lengths truncated at the upper limit by the domain extent and at the lower limit by the spacing of the finite-difference grid.

Apparent Flow Dimension (Steps 2 and 3)

For the estimation of the apparent flow dimension (Step 3), a constant-rate transient aquifer test was simulated (Step 2) in one realization of the transmissivity field created in Step 1. The well was assigned to the central node of the grid and constant-head boundary conditions were imposed on the four sides of the model domain, which was large enough to avoid boundary effects. A simple finite-difference approximation in time was used to evaluate Eq. [4], and the apparent flow dimension was evaluated from Eq. [3]. The aquifer test was repeated for multiple Monte Carlo realizations of the transmissivity field, and the mean and 95% confidence intervals of the apparent flow dimensions, \( n^* \), were used to summarize the ensemble of realizations.

Anomalous Diffusion Coefficient (Step 4)

Researchers have often studied radial diffusion using the analogous model of a random walk followed by particles released from a central location. In such random walk models, the diffusion process is characterized by the time rate of change of the mean square radius of displacement \( \langle R^2 \rangle \) of the particles (Saadatfar and Sahimi, 2002). To help relate the results of random walk analog models to the results of this research for the flow dimension, this study estimated \( \langle R^2 \rangle \) at each time step by counting the model cells with a drawdown larger than a prescribed tolerance, \( s_{\text{tol}} \). The total area occupied by these cells was set equal to the area of a circle centered at the pumped well, and the squared radius of that circle was the estimate of \( \langle R^2 \rangle \) for that time step. The estimate of \( \langle R^2 \rangle \) is the arithmetic mean of \( R^2 \) averaged across the Monte Carlo realizations. Cells representing the low-permeability matrix between the fractures were included in the estimate if their drawdown was larger than \( s_{\text{tol}} \) because they could still contribute to the diffusion of the system. The prescribed drawdown tolerance was at least one order of magnitude larger than the convergence tolerance for the numerical solver in MODFLOW, and was the same for all models.

The drawdown tolerance, \( s_{\text{tol}} \), was chosen by noting that the constant-head model boundaries were just beginning to contribute flow at the final time step of the simulation; that is, the influence of the aquifer test had just reached the limits of the model. The tolerance, then, is the drawdown that is observed near the model boundaries when the boundaries just begin to contribute flow. Consequently, the mean radius of displacement \( R \) is always less than the radial distance from the pumped well to the model boundaries. This approach was used rather than an analytical solution for the radius of influence (e.g., Oliver, 1990), which requires effective values for transmissivity and storativity that are not easily defined for highly heterogeneous media. The apparent exponent \( k(n)^* \) was obtained by finite-difference approximation in time for the slope of the logarithmic transform of the scaling law given by Eq. [6].

Fractal Dimension Analysis (Step 5)

It is common in many studies to assess the geometry of a fracture system by computing \( D \), the mass fractal dimension of the fractures (Bonnet et al., 2001). In our study, we evaluated \( D \) (Eq. [5]) at different length scales of the aquifer test and explored its relation to the apparent flow dimension and the apparent exponent of diffusion. The circles of investigations were centered at the pumping well location and the radii were chosen equal to the value of \( R \) estimated in Step 4. The mass of fractures \( M(R) \) was estimated by counting the model cells representing a fracture within radius \( R \) of the well.

Model Results

The multiscale nature of fracture networks requires large networks to be simulated with a fine resolution. The finite difference grid used in this study had a uniform 1-m spacing to ensure that the stochastic models were accurately represented, while the model domain was extensive (3001 by 3001 nodes) to permit simulating aquifer tests without boundary effects. The simulated aquifer is consistent with our prior work (Walker et al., 2006b) and was loosely based on the Culebra dolomite, a fractured dolomite aquifer found at the Waste Isolation Pilot Plant (WIPP) near Carlsbad, NM (Holt, 1997). A well withdrawing at a constant rate of \( 2.28 \times 10^4 \text{ m}^3/\text{s} \) was assigned to the central node of the grid, and constant-head boundaries were imposed on the four sides of the model domain. In all simulations, the flow balance error of MODFLOW-2000 was less than ±0.05% of the outflow or inflow (see Walker et al., 2006a, for additional details).

This work aimed to infer the average behavior of fluid flow and the pressure transient diffusion of a fracture network with fractal characteristics. Therefore, the arithmetic means and normal confidence intervals of the apparent flow dimension, the apparent exponent \( k(n)^* \), and the mass fractal dimension were computed among the Monte Carlo realizations. Walker et al. (2006b) found that 200 Monte Carlo realizations of both site percolation network (SPN) models and mvG models were sufficient for stable estimates of the mean and variance of the apparent flow...
dimensions. Since the apparent flow dimensions of the DFN model show less variability among realizations than do realizations of the SPN, we expected that 200 realizations would be sufficient for stable estimates comparable to the literature for this study.

Multivariate Gaussian Models

The calculations were repeated for log transmissivity distributed as a mvG model to verify the approach and to provide a comparison case. For this study, the mvG field was created using the sequential Gaussian simulation algorithm of Deutsch and Journel (1998) with an exponential semivariogram, an integral scale I = 7 m, and a variance of $\sigma_{\text{ln}}^2 = 1$ (Fig. 2a). A single conditioning value was located at the pumped well with a transmissivity of $T_w = 4.7 \times 10^{-5}$ m²/s, the geometric mean observed at the WIPP site. The aquifer test simulation consisted of a single transient stress period of 345,600 s, 44 time steps, a time-step multiplier of 1.3, and initial elapsed time of 1 s.

The impact of finiteness of the domain was monitored using constant-head boundaries; as the cone of depression expands, the model calculates the flow induced from these constant-head nodes. The total flow from the exterior constant-head boundaries relative to the well pumping rate was in no case greater than the flow balance error of the model to limit boundary effects. Figure 2c shows the impact of encountering the boundaries as an upward deflection of the estimated flow dimension and exponent of diffusion at the last two time steps of the aquifer test. The time when this occurred depended on the parameters of each case, thus the duration of each of the simulated cases varied slightly.

Figure 2c shows that for a moderate variance, the arithmetic average for 200 realizations of the apparent flow dimension converged rapidly to $n^* = 2$. The variability between realizations decreased with time, suggesting that even individual aquifer tests in a mvG field will tend to show a radial flow pattern with $n^* = 2$ (Fig. 2b) if the radius of investigation is much greater than the integral scale. Walker et al. (2006a) showed that this tendency is also true for mvG models of larger variances. As the apparent exponent rapidly converges to $k^* = 1$, its variance among realizations decreases with time (Fig. 2c). The apparent flow dimension reflects the radial geometry of flow and the hydraulic diffusion is apparently Fickian, in agreement with the literature for the mvG case.

Discrete Fracture Network Model

The DFN model was simulated using a modified version of the Boolean simulation algorithm of Deutsch and Journel (1998). Fracture lengths were distributed according to a power-law model using three different values of the exponent $a$. The lower truncation limit for fracture length was set at 1 m, equal to the spacing of the model grid, and an extended virtual domain of 4500 m (for an upper limit on length of 4500 m) was used to avoid edge effects in the simulated fracture network.

This study was focused on DFN cases where the flow dimension stabilized to a noninteger value for much of the test, similar to flow dimensions observed during aquifer tests in the field. Thus, two different values of fracture intensity, $p$ (the number of fracture cells per unit area), were chosen large enough to guarantee that the cluster of fractures containing the pumped well spanned the entire domain, just as well driller and reservoir engineers seek—and often force—wells that connect to extensive, productive fracture networks.

Two fracture sets were generated such that they intersected at right angles, similar to fracture patterns in dolomite observed at other sites (Foote, 1982; Roffers, 1996). The cell representing the pumped well was always represented by a fracture. The transmissivity of the fracture cells was $T_f = 4.7 \times 10^{-5}$ m²/s and matrix cells were assigned $T_m = 4.7 \times 10^{-9}$ m²/s (i.e., the matrix was less transmissive by a factor of 1/10,000). The aquifer test simulation consisted of a single transient stress period of 1,283,179 s, 49 time steps, and a time-step multiplier of 1.3. For the estimation of $R^2$, the drawdown tolerance was set to $s_{\text{tot}} = 0.025$ m.

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**Fig. 2.** Results for the multivariate Gaussian model: (a) one realization of the log transmissivity field (integral scale of 7 m and variance of 1.0); (b) one realization of drawdown at elapsed time $t = 7 \times 10^4$ s; (c) mean and 95% confidence interval for 200 realizations of the apparent flow dimension ($n^*$) and the apparent diffusion coefficient ($k^*$); $D_w$ is the fractal dimension of the random walk.
The influence of fracture connectivity on the apparent values of the flow dimension (Eq. [3]), diffusion exponent (Eq. [6]), and mass fractal dimension (Eq. [5]) was examined using four cases: two cases in which the exponent \(a\) changed while the intensity \(p\) was held constant and two cases where the intensity changed while the exponent was held constant. Table 1 summarizes the statistics of the fracture lengths for the four different cases. In general, decreasing the exponent \(a\) for a constant intensity parameter increased the frequency of longer fractures, thus fewer fractures were necessary to create a connected network that spanned the domain. Increasing \(a\) with a constant intensity decreased the probability of having large fractures and the connectivity was carried by a larger number of smaller fractures. Consequently, as the exponent \(a\) increased, the mean and median fracture length in the domain tended to decrease and the proportion of fracture lengths smaller than the size of the domain tended to increase (Table 1). This effect is clearly seen in the increased sparseness of the fracture network as \(a\) increased from 1.2 to 2.0 (compare Fig. 3a and 4a). The same trend can be observed in comparing Fig. 5a and 6a, where \(a\) increased from 2.0 to 2.5. Although both small and large fractures were connected to the flowing cluster, the largest fractures controlled the connectivity for \(a = 1.2\) (Case 1) and the smaller fractures controlled the connectivity when \(a\) increased to 2.5 (Case 4), as noted by Bour and Davy (1997).

Before presenting the detailed results from the Monte Carlo simulations, it is instructive to discuss the expected asymptotic behavior of the flow dimension and the anomalous diffusion coefficient at the limits of the connectivity regimes when \(a = 1\) and 3. When the exponent \(a\) of the power-law model tends to 1, the connectivity of a fracture network at an intensity \(p\) equal to that of the percolation threshold is reached by a few long fractures. For that condition, we would expect to have a flow dimension \(n^*\) that tends to 1 (reflecting the pipe-like geometry of fluid flow [Barker, 1988]), the fractal dimension of the network would tend to \(D = 1\), and pressure transients would diffuse as a Fickian process. At the other extreme where exponent \(a\) approaches 3, the connectivity of a fracture network near the percolation threshold arises from a large number of shorter fractures. For this condition, as the hydraulic...
Therefore, when the exponent \( \alpha \) increases, we expect increases in the flow dimension, the exponent of diffusion, and the fractal dimension (if the fracture system network is near the percolation threshold). The percolation threshold can be explained as the minimum proportion of fracture cells that allow the cluster to span the entire model domain (Balberg, 1986); moreover, it was demonstrated that for a constant model size, the percolation threshold should increase as the exponent \( \alpha \) increases (Bour and Davy, 1997). Therefore, for a constant intensity parameter, increasing \( \alpha \) will reduce the number of connected paths but will increase the proportion of dead-end fractures; together these tend to oppose any increase in the flow dimension and the exponent of diffusion.

Table 2 summarizes the effect of changing the exponent \( \alpha \) for a constant intensity \( p \) and vice versa. This table reports the stable values of flow dimension \( n^* \) and exponent of diffusion \( k^* \) averaged across Monte Carlo realizations within a late time–space range of the aquifer test. Ranges of the local fractal dimension \( D_R \) within the same temporal–space frame are also included. Stable values of \( n^* \) and \( k^* \) reported in this table were computed using Eq. [3] and [6], respectively, and averaged across the stable time–space range. In addition, the reported range of local fractal dimensions \( D_R \) averaged across Monte Carlo realizations were computed using Eq. [5] within the same time–space range. The time–space ranges were obtained by inspection. In general, the sensitivity to changing the exponent \( \alpha \) also depends on the magnitude of the intensity parameter. The effect of increasing the exponent \( \alpha \) on the flow dimension and the anomalous diffusion coefficient (increasing \( n^* \) and \( k^* \)) is counterbalanced by the effect of a constant intensity (see Cases 3 and 4 in particular in Table 2). Increasing the intensity \( p \) while maintaining the exponent \( \alpha \) tends to increase the flow dimension, the anomalous diffusivity coefficient, and the fractal dimension (see Table 2).

Figure 3c illustrates the results for Case 1 \((\alpha = 1.2, p = 0.25)\). The apparent flow dimension converged to a noninteger value \(<2.0\) approximately at time \( t = 5.2 \times 10^4 \) s and remained stable for almost 1.5 log cycles of the aquifer test. At later times and larger scales of the aquifer test, the apparent flow dimension tended to steadily increase to 2.0 (Table 2). The exponent for the scaling of diffusion also converged to \( k^* \) close to 1 at later time and remained approximately stable even for temporal–spatial scales larger than in the case of the flow dimension (Table 2). The mass fractal dimension steadily increased with time to a value close to 2. Figures 3b and 3c show that the drawdown started to show radial geometry with local dimensions \( D_R \) between 1.81 and 1.90 for a period between \( 1.9 \times 10^4 \) and \( 3.5 \times 10^5 \) s.

Case 2 \((\alpha = 2.0, p = 0.25)\) examined the effects of changing the exponent \( \alpha \) while keeping the intensity parameter \( p \) unchanged (Fig. 4c). Increasing \( \alpha \) from 1.2 to 2 resulted in an increase in both the apparent flow dimension and the apparent anomalous diffusion coefficient (Table 2). Both the apparent flow dimension and the apparent anomalous diffusion coefficient converged to stable values and remained approximately constant for larger spatial scales than in Case 1 (see Fig. 4c and Table 2). We attribute this to maintaining the intensity parameter \( p \) with a consequent reduction in the connected paths of the fracture network of Case 2, where percolation theory suggests that the correlation length of the cluster should increase because the intensity is closer to the percolation threshold than in Case 1 (Sahimi, 1995; Stauffer and Aharony, 1994). Consequently, we expected a fractal behavior for longer spatial scales of the aquifer test in
Case 2 (Table 2). Increasing $a$ tended to increase the mass fractal dimension, but this effect was also counterbalanced by the effect of maintaining the intensity $p$. Figures 4b and 4c show that the mass fractal dimension $D_R$ was in a range between about 1.82 and 1.88 for a period between $2.5 \times 10^4$ and $5.8 \times 10^5$ s.

The next variant of the DFN model, Case 3 ($a = 2.0, p = 0.35$), shows the effect of changing the intensity parameter for a constant exponent $a$ (Fig. 5c). Here, increasing the intensity tended to increase the apparent flow dimension, the apparent exponent of diffusion, and the mass fractal dimension. While the apparent flow dimension steadily tended to 2.0, the exponent of diffusion converged to $k^* = 0.93$. The spatial scale at which the system started to behave as macroscopically homogeneous ($n^*$ tended to 2.0) and Fickian ($k^*$ tended to 1.0) was smaller than in the previous case (Table 2). Comparing Fig. 4b and 5b, the pressure transients travelled faster when the intensity parameter was increased. Figure 5c shows that the mass fractal dimension $D_R$ was within the range of 1.88 to 1.92 for a period of between $1.5 \times 10^4$ and $2.0 \times 10^5$ s and steadily increased to 2.0.

Case 4 increased $a$ from 2.0 to 2.5 while maintaining $p = 0.35$. Contrary to Case 1, smaller fractures controlled the connectivity of the network (Bour and Davy, 1997). The apparent flow dimension and the apparent anomalous diffusion coefficient converged to $n^* = 1.78$ and $k^* = 0.88$, respectively (Fig. 6c). The apparent flow dimension converged to a stable value at a later time and for a shorter period. On the other hand, the apparent exponent for the scaling of diffusion stabilized at an earlier time scale and remained at an approximately constant value until the end of the aquifer test (see Fig. 6c and Table 2). The mass fractal dimensions steadily increased with spatial scale, with magnitudes slightly larger than those estimated in the previous case. The behaviors of the flow dimension, the anomalous diffusion coefficient, and local fractal dimension between Cases 3 and 4 were opposite to those observed between Cases 1 and 2. We speculate that this is because of maintaining the intensity $p$ while increasing the exponent.

**Table 2. Summary of model results for a discrete fracture network model at different connectivity regimes.**

<table>
<thead>
<tr>
<th>Statistic</th>
<th>Case 1: $a = 1.2, p = 0.25$</th>
<th>Case 2: $a = 2.0, p = 0.25$</th>
<th>Case 3: $a = 2.0, p = 0.35$</th>
<th>Case 4: $a = 2.5, p = 0.35$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stable apparent flow dimension ($n^*$)</td>
<td>1.86</td>
<td>1.89</td>
<td>1.92 (tended to 2)</td>
<td>1.78</td>
</tr>
<tr>
<td>Time range for stable ($n^*$), s†</td>
<td>$5.2 \times 10^3$–$1.6 \times 10^5$</td>
<td>$5.2 \times 10^3$–$2.7 \times 10^5$</td>
<td>$5.2 \times 10^3$–$2.7 \times 10^5$</td>
<td>$6.7 \times 10^3$–$1.6 \times 10^5$</td>
</tr>
<tr>
<td>Space range for stable ($n^*$), m†</td>
<td>116–593</td>
<td>106–656</td>
<td>121–755</td>
<td>91–375</td>
</tr>
<tr>
<td>Local fractal dimension ($D_w$) range for stable ($n^*$)</td>
<td>1.75–1.89</td>
<td>1.77–1.87</td>
<td>1.86–1.92</td>
<td>1.84–1.91</td>
</tr>
<tr>
<td>Stable apparent diffusion coefficient ($k^*$), m²/s</td>
<td>0.86</td>
<td>0.91</td>
<td>0.93</td>
<td>0.88</td>
</tr>
<tr>
<td>Time range for stable ($k^*$), s†</td>
<td>$1.9 \times 10^4$–$3.5 \times 10^5$</td>
<td>$2.5 \times 10^4$–$5.8 \times 10^5$</td>
<td>$1.5 \times 10^4$–$2 \times 10^5$</td>
<td>$2.5 \times 10^4$–$1 \times 10^6$</td>
</tr>
<tr>
<td>Space range for stable ($k^*$), m†</td>
<td>220–862</td>
<td>225–941</td>
<td>199–668</td>
<td>165–860</td>
</tr>
<tr>
<td>Local fractal dimension ($D_R$) range for stable ($k^*$)</td>
<td>1.81–1.90</td>
<td>1.82–1.88</td>
<td>1.88–1.92</td>
<td>1.88–1.92</td>
</tr>
</tbody>
</table>

† Values obtained by inspection.
In addition, Fig. 6b and 6c show that the mass fractal dimension was between 1.88 and 1.92 for a period between $2.5 \times 10^4$ and $1 \times 10^6$ s.

Figures 3b to 6b in combination with Fig. 7 also demonstrate how the mean radii of inspection of the aquifer test propagate with different speeds depending on the chosen DFN model parameters. Increasing the exponent $a$ while maintaining the intensity $p$ (or vice versa) makes pressure transients propagate at a slower speed. Consequently, the spatial scale of the aquifer test beyond which the system behaved macroscopically homogeneous and Fickian seems to depend also on the connectivity regime of the fracture network, and, hence, on both the intensity parameter and the exponent of the power-law model (Table 2).

The variability of the apparent flow dimension and exponent of diffusion among realizations generally decreased with time, but the late-time behavior of the diffusion coefficient was less variable than that of the flow dimension. Increasing $a$, the slope of the power law, for the same intensity increased the variability among realizations of both the flow dimension and the exponent of diffusion. The same effect can be achieved by decreasing the intensity for the same value of $a$. For Case 4, decreasing the power-law slope $a$ reduced the number of long fractures and consequently reduced the connectivity. This reduction in connectivity tended to increase the variability among realizations, thus widening the confidence intervals of both the flow dimension and the exponent of diffusion. The same effect can be achieved by decreasing the intensity for the same value of $a$.

Summary and Conclusions

This study examined the behavior of the apparent flow dimension and the exponent of diffusion for a DFN model for different connectivity regimes controlled by the intensity parameter $p$ and the exponent $a$ of the power-law model for fracture lengths. In particular, we analyzed whether noninteger flow dimensions are associated with anomalously slow diffusion and how these are related to the connectivity and fractal geometry of the fracture network. The flow dimensions and the exponent of diffusion for a DFN model also were compared with those obtained for a mvG model to show that the behavior of fluid flow and transport can differ depending on the chosen stochastic model of aquifer heterogeneity.

Figure 7 summarizes the results plotted vs. the radius of influence of the aquifer test for the mean apparent flow dimension and the mean apparent exponent of diffusion. The mean values are for 200 realizations of the DFN model at different connectivity regimes and of the mvG model. The apparent flow dimensions and the apparent diffusion exponents of the mvG model appear to stabilize to approximately $n^* = 2$, while those of the DFN appear to stabilize to $n^* < 2$ with a magnitude that depends on both the exponent $a$ of the
power-law length distribution and the intensity parameter $p$. Increasing the exponent of the power-law model while maintaining the intensity parameter value reduces the connectivity of the fracture network and decreases the apparent flow dimension and the anomalous diffusion coefficients. Reducing the intensity while maintaining the exponent $a$ also reduces the connectivity of the fracture network and reduces both the flow dimension and the anomalous diffusion coefficient. Besides, for the DFN model, while the flow dimension stabilizes to a value less than the Euclidean dimension at late time of the aquifer test and for a particular connectivity regime, the anomalous diffusion coefficient confirms the non-Fickian behavior of pressure transient diffusion. The sensitivity of both the flow dimension and the diffusion coefficient to model parameters suggests that they might be useful as indicators of connectivity regimes in fractured media and thus improve the characterization and modeling of fluid flow and pressure transient diffusion in fractured media.

ACKNOWLEDGMENTS

This study was sponsored by the Illinois Water Resources Center under the NIWR 104G program, Illinois State Water Survey (ISWS) general revenue funds, Sandia National Laboratories (SNL) under SNL PO no. 246992, the National Center for Supercomputing Applications, and the National Science Foundation. The views expressed are those of the authors and do not necessarily reflect on the view of the sponsors or the ISWS. We acknowledge the assistance of Steffan Mehl (USGS); Rick Bauheim, Glenn Hammond, Sean McKenna, and Randy Roberts (SNL); and Scott Meyer and H. Allen Wehrmann (ISWS) for discussion and constructive criticism on this paper.

References


![Figure 7](image-url) Mean apparent flow dimension ($n^*$) and mean anomalous diffusion coefficient ($k^*$) for 200 realizations as a function of the mean radius of investigation of the aquifer test using the discrete fracture network model for different connectivity regimes (Cases 1–4 with different exponents $a$ and intensities $p$) and the multivariate Gaussian (mvG) model with integral scale $l = 7 m$ and variance $\sigma_{\ln}^2 = 1.0$; $D_w$ is the fractal dimension of the random walk.