Flow dimensions corresponding to hydrogeologic conditions

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Received 10 June 2002; revised 24 April 2003; accepted 20 August 2003; published 17 December 2003.

[1] The generalized radial flow approach to hydraulic test interpretation uses the flow dimension to describe the change in flow area versus radial distance from the borehole. The flow dimension of a hydraulic test may reflect several characteristics of the hydrogeologic system, including heterogeneity, boundaries, and leakage. We show that a radial flow system with a linear, constant-head boundary asymptotically reaches a flow dimension of four, while the flow dimension of a leaky aquifer is a function of time and the leakage factor. We use numerical techniques to show that a stationary transmissivity field with a modest level of heterogeneity has a flow dimension that stabilizes at two. We also show that the flow dimension for a nonstationary transmissivity field depends on the form and magnitude of the nonstationarity. The flow dimension observed during hydraulic tests helps identify admissible conceptual models for the tested system, and places hydraulic testing in its full hydrogeologic context. *INDEX TERMS:* 1829 Hydrology: Groundwater hydrology; 1894 Hydrology: Instruments and techniques; 5114 Physical Properties of Rocks: Permeability and porosity; *KEYWORDS:* hydraulic test, generalized radial flow, flow dimension, heterogeneity, aquifer test

Citation: Walker, D. D., and R. M. Roberts, Flow dimensions corresponding to hydrogeologic conditions, *Water Resour. Res.*, *39*(12), 1349, doi:10.1029/2002WR001511, 2003.

1. Introduction

[2] A fundamental tool of applied hydrogeology is hydraulic testing in boreholes to determine the formation hydraulic properties, evaluate the formation extent, and assess leakage. These tests consist of observing the propagation of a pressure perturbation in the formation of interest and interpreting the test by fitting the observations with an idealized flow model. Interpretive models commonly assume an idealized geometry in homogeneous, infinite domains, e.g., the linear flow model of *Miller* [1962] and the radial flow model of *Theis* [1935]. These interpretive models have proven useful in characterizing aquifers and petroleum reservoirs, but many formations violate the assumptions of simple geometry and homogeneous properties. When the tested formation deviates from the idealized formation of the interpretive model, an alternative method of test interpretation is the generalized radial flow (GRF) approach of *Barker* [1988]. The GRF approach considers systems where the relationship between cross-sectional area of flow and distance from the source is given by

$$A(r) = \alpha_n r^{n-1}, \qquad (1)$$

where the surface area of a unit sphere in n dimensions is

$$\alpha_n = b^{3-n} \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \qquad (2)$$

and where $A(r)$ is the cross-sectional area of flow [L^2], r is the radial distance from the borehole [L], b is the extent of the flow zone [L], n is the flow dimension [ ], and $\Gamma(\,)$ is the gamma function [ ]. The flow dimension n describes the geometry of the system by defining the rate that the cross-sectional flow area changes with respect to distance from the test well, i.e., the flow dimension is the power by which the flow area changes with respect to radial distance, plus one. For example, in a homogeneous, radial system the cross-sectional area of flow at any distance r is proportional to r^1, i.e., $A(r) = 2\pi rb$, so that the flow dimension is $n = 2$.

[3] The GRF model is not restricted to flow dimensions of 1, 2, or 3 (i.e., linear, radial, or spherical flow), nor is a real hydrogeologic system. Consider, for example, a domain with a Euclidean dimension of two where heterogeneities restrict the flow such that the cross-sectional area of flow expands proportional to $r^{0.7}$; in such circumstances it is unreasonable to expect that the commonly assumed radial ($n = 2$) interpretive model would adequately describe a test in the domain. The GRF approach has been used to interpret hydraulic tests in a variety of settings [e.g., *Bangoy et al.*, 1992; *Doe and Geier*, 1991], but few solutions have been presented for the flow dimensions corresponding to specific hydrogeologic systems.

[4] This paper presents solutions for the flow dimensions that correspond to idealized hydrogeologic systems, with the objective of aiding the interpretation of the flow dimensions of hydraulic test responses. These solutions are developed for constant-rate hydraulic tests with a fully penetrating source in a domain with a Euclidean dimension of two. We summarize the proofs for the known flow dimensions of the Theis problem with and without a linear

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no-flow boundary. We derive analytical solutions for the flow dimension of the Theis problem with a linear, constant-head boundary and for the Hantush leaky aquifer problem. We derive analytical solutions for the flow dimensions of fields with specific forms of nonstationary transmissivity and approximate the flow dimension of a stationary transmissivity field with periodic heterogeneity.

2. Background

[5] *Barker* [1988] developed the generalized radial flow (GRF) approach for well test interpretation to address hydraulic tests in fractured rocks, where flow does not necessarily fill the tested domain. The assumption that the cross-sectional area of flow is given by equation (1) leads to the GRF governing equation:

$$S_s \frac{\partial h}{\partial t} = \frac{K}{r^{n-1}} \frac{\partial}{\partial r}\left(r^{n-1} \frac{\partial h}{\partial r}\right) \tag{3}$$

where $h(r, t)$ is the change in hydraulic head from $h(r, 0) = 0$ [L], K is the hydraulic conductivity [L/T], and S_s is the specific storage [1/L]. Barker showed that solutions could be obtained for specific boundary conditions using integral transforms. For example, for a constant discharge Q_0 [L^3/T] from a point source in an infinite domain with $h(r, 0) = 0$, Barker found the generalized solution:

$$h(r,t) = \frac{Q_0 r^{2v}}{4\pi^{1-v} K b^{3-n}} \Gamma(-v, u) \tag{4}$$

where

$$u = \frac{r^2 S_s}{4Kt} \tag{5}$$

$$v = 1 - n/2. \tag{6}$$

Depending on the value of the flow dimension, equation (4) can be reduced to a particular solution for $h(r, t)$ and fitted to observed pressure changes by varying the hydraulic parameters K and S_s. For example, radial flow in an infinite, homogeneous domain with a Euclidean dimension of two is defined as $n = 2$, so that equation (4) reduces to the solution of *Theis* [1935]:

$$h(r,t) = \frac{Q_0}{4\pi K b} W(u) \tag{7}$$

where $W(u) = E_1(u)$ is the exponential integral, also known as the Theis well function:

$$W(u) = \int_u^\infty \frac{e^{-y}}{y} dy. \tag{8}$$

[6] Our discussion thus far has assumed that the flow dimension of the tested system is known a priori, interpreting the hydraulic test by fitting a model based on the assumed flow dimension. More generally, however, the GRF approach interprets a hydraulic test by varying the flow dimension, as well as the K and S_s, until the GRF model fits the test response. The flow dimension need not be the same as the Euclidean dimension of the tested system, as this paper will show.

[7] One cause for the flow dimension differing from the Euclidean dimension of the formation is that impermeable boundaries have restricted the area of flow. For example, a hydraulic test in an aquifer with a Euclidean dimension of two that is bounded by parallel, linear, no-flow boundaries can transition from radial flow ($n = 2$) to linear flow ($n = 1$) as the boundaries restrict the flow. This geometric interpretation of the flow dimension is easily extended to the case of flow in fractured rocks, where flow is restricted to some portion of the domain. *Barker* [1988] conjectured that noninteger values of flow dimensions observed in fractured systems might be the result of radial diffusion in a fractal network of fractures. *Acuna and Yortsos* [1995] found that under certain conditions, the fractal dimension of a fracture network equals the flow dimension and that the flow dimension need not be an integer. The flow dimension reflects how the flow area apparently changes with distance from the source and is not necessarily related to the space-filling nature of the flow. *Barker* [1988] noted that a single pipe-like conduit could follow a tortuous path, effectively filling a three-dimensional domain, and yet still have a flow dimension $n = 1$.

[8] The flow dimension also reflects heterogeneous aquifer properties; *Doe* [1991] pointed out that flow geometry and heterogeneity are, in fact, continuously interchangeable as interpretations of the flow dimension. This can be seen if we rederive the GRF governing equation for the case of heterogeneous properties, $K = K(r)$ and $S_s = S_s(r)$. Similar to *Barker* [1988], we combine the Taylor's series expansion of the continuity equation with Darcy's law to find

$$S_s(r) \frac{\partial h}{\partial t} = \frac{1}{r^{n-1}} \frac{\partial}{\partial r}\left(K(r) r^{n-1} \frac{\partial h}{\partial r}\right) \tag{9}$$

For flow to a well in a system with a Euclidean dimension of 2 and with homogeneous properties $K(r) = K_o$ and $S_s(r) = S_{so}$, equation (9) reduces to the governing equation for radial flow, $n = 2$:

$$S_{so} \frac{\partial h}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r}\left(K_o r \frac{\partial h}{\partial r}\right) \tag{10}$$

However, suppose that the hydraulic properties are heterogeneous with $K(r) = K_o r$ and $S_s(r) = S_{so} r$; in this case, equation (9) reduces to

$$S_{so} \frac{\partial h}{\partial t} = \frac{1}{r^2} \frac{\partial}{\partial r}\left(K_o r^2 \frac{\partial h}{\partial r}\right) \tag{11}$$

Equation (11) is identical to the governing equation for spherical flow with homogeneous properties, and thus a hydraulic test in this system would have a flow dimension of $n = 3$. *Carslaw and Jaeger* [1959] presented a general set of solutions for equation (9) where hydraulic properties vary as a power of distance from the source term, and they noted the interchangeability of solutions for linear, radial, and spherical systems. *Doe* [1991] noted that this interchangeability means that the flow dimension interpreted for a

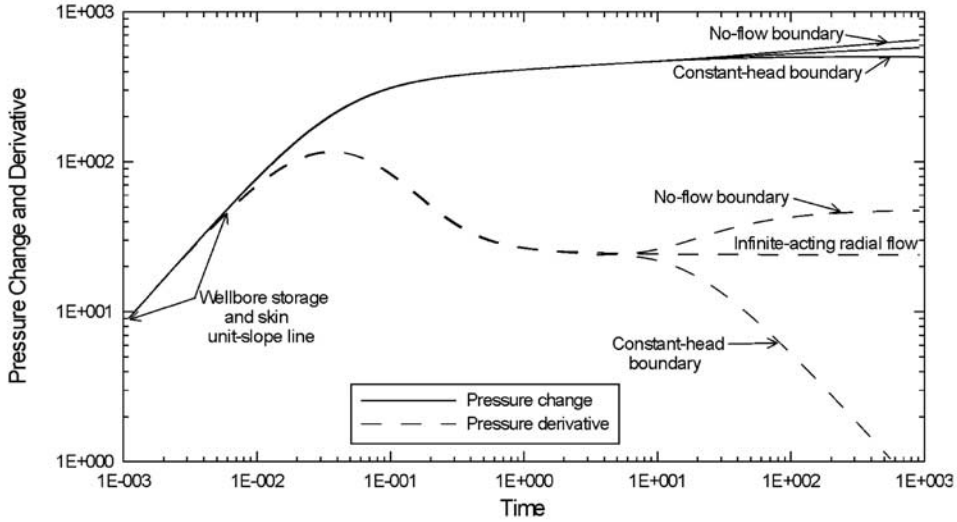

Figure 1. Log-Log diagnostic plot of pressure change and the derivative of the pressure change at the pumped well for a constant-rate test, showing responses of idealized models for infinite-acting radial flow, linear no-flow and linear constant-head boundaries.

hydraulic test might be the consequence of heterogeneity, variations in geometry, or some combination of both. This implies that the observed flow dimension is not necessarily a simple function of radial distance as it is in the homogeneous systems described by *Barker* [1988]. Given the multiple interpretations of an observed flow dimension, it is reasonable to ask what flow dimensions correspond to specific hydrogeologic conditions and what are possible interpretations for seemingly nonintuitive values of the flow dimension.

[9] A convenient starting point for this analysis is the log-log diagnostic plot developed by *Bourdet et al.* [1983a] for a constant-rate hydraulic test. Figure 1 provides a stereotypical example of this diagnostic plot using arbitrary parameters, where the common logarithm of pressure change is plotted versus the common logarithm of elapsed time, together with $\log(dh/d\ln t)$, the common logarithm of the derivative of pressure change with respect to the natural logarithm of time. The plot of $\log(dh/d\ln t)$ versus the common logarithm of time is known as the pressure derivative, and its characteristics are unique for each value of the flow dimension. *Ehlig-Economides* [1988], among others, noted that the pressure derivative displays straight lines at late time, with slopes related to the domain dimensionality and boundary condition. *Mishra* [1992] showed that the late-time slope of the pressure derivative for an infinite-acting system is related to the GRF model by

$$\nu = \lim_{t\to\infty} \frac{d}{d(\log t)} \left[\log\left({}^{dh}/_{d\ln t}\right)\right] \tag{12}$$

and, rearranging equation (6), the flow dimension can be found as $n = 2 - 2 \cdot \nu$. For example, infinite-acting radial flow has a flow dimension of $n = 2$ by definition, so that the late-time slope of its pressure derivative is $\nu = 1 - 2/2 = 0$. Equations (6) and (12) thus provide a simple means for determining the flow dimension for idealized hydrogeologic conditions from the late-time slope of the pressure derivative. For some systems, the slope of the pressure derivative may initially stabilize at one value before reaching a different value at the limit described by equation (12), such that the flow dimension will have apparent values prior to reaching its asymptotic limit. For the purposes of this paper, we will plot the apparent flow dimension, $2 - 2 \cdot \nu^*$, versus dimensionless time $t_D = 1/u$, where the slope of the pressure derivative is given by

$$\nu^* = \frac{d}{d(\log t)} \left[\log\left({}^{dh}/_{d\ln t}\right)\right] \tag{13}$$

[10] One approach to determining the slope of the pressure derivative is to numerically differentiate an appropriate analytical solution. This numerical approach has been used to determine the late-time slopes for the pressure derivative of dual porosity systems [*Bourdet et al.*, 1983b]; hydraulically fractured wells [*Ehlig-Economides*, 1988]; radial composite systems [*Butler*, 1988]; linear composite systems [*Butler and Liu*, 1991]; and for a closed outer boundary [*Ehlig-Economides*, 1988]. *Ehlig-Economides* [1988] and *Horne* [1995], among others, used numerical differentiation of superimposed solutions to determine the late-time slope of the pressure derivatives of various combinations of linear boundaries. In some cases, it is possible to directly differentiate an analytical solution for $h(r, t)$ in the hydrogeologic system of interest.

[11] In the following sections, solutions are developed for the slope of the pressure derivatives of a hydraulic test in specific hydrogeologic systems and thus their corresponding apparent flow dimensions. The asymptotic values of the flow dimension are determined via equation (12), where they exist. All cases assume a constant-rate hydraulic test in a domain with a Euclidean dimension of two and a fully penetrating well. Table 1 presents the values of the parameters used in the examples for each case. We first provide a proof for the flow dimension of infinite-acting

Table 1. Values of Parameters Used in the Examples (Figure 2)

Parameter	Symbol and Value
Hydraulic conductivity of aquifer	K = 1e −2 m/s
Thickness of aquifer	b = 10 m
Storage coefficient of aquifer	S = 1e − 3 [ ]
Pumping rate (negative for image well in the linear no-flow boundary case)	Q = 3.15e − 2 m3/s
Radius of the pumped well (and the observation radius)	r_w = 0.3 m
Radial distance from the pumped well to the image well	r_i = 30 m
Hydraulic conductivity of confining unit	K' = 1e − 4 m/s
Distance from the overlying water table to the average head level of the aquifer	b' = 5 m
Leakage factor	B = 71 m
Parameter controlling thickness change	a = 71 m

radial flow in a homogeneous domain, developed by differentiation of the analytical solutions. This is extended to the cases of simple linear boundaries and leakage. Exact solutions are derived for the flow dimension for certain cases of nonstationary heterogeneous transmissivity. An approximate solution is developed for a stationary heterogeneous transmissivity of modest variability, based on numerical differentiation of a semianalytical solution.

3. Exact Solutions for Radial Flow

[12] We first examine the flow dimension for a hydraulic test in an infinite, homogeneous, radial system. Although the flow dimension of such a system is by definition $n = 2$, its proof provides a strategy for deriving the flow dimension of more complex systems. We begin by noting that the analytical solution for $h(r, t)$ for a constant-rate hydraulic test in this system is the solution of *Theis* [1935], previously given as equation (7). Substituting equation (8) into equation (7) and noting that the pressure derivative is $dh/d\ln t = t dh/dt$, we get

$$\frac{dh}{d\ln t} = t\frac{Q_0}{4\pi Kb}\frac{d}{dt}\left[\int_u^\infty \frac{e^{-y}}{y}dy\right], \tag{14}$$

where u is defined as before. Applying Leibniz' rule to equation (14) yields

$$\frac{dh}{d\ln t} = \frac{Q_0}{4\pi Kb}\exp[-u] \tag{15}$$

Substituting equation (15) into equation (12), and reducing to find the late-time slope of the pressure derivative:

$$v_{\text{Theis}} = \lim_{t\to\infty} u. \tag{16}$$

Equation (16) confirms the more general solution for the slope of the pressure derivative found by *Mishra* [1992]. Evaluating the limit of equation 16 yields $v_{\text{Theis}} = 0$, and rearranging equation (6) and substituting, we get $n = 2 - 2 \cdot 0 = 2$. The flow dimension for radial flow in an infinite, homogeneous domain is thus proven to be $n = 2$, as defined. This flow dimension is reached when $t \gg r^2S_s/4K$ (Figure 2).

[13] The preceding strategy may be used to derive the flow dimension for a linear, no-flow boundary imposed on radial flow. The familiar analytical solution for $h(r, t)$ in this case is the superposition of solutions for the well and its image reflected across the boundary, both represented by equation (7) [*Ferris et al.*, 1962]. Applying the above strategy to the superposed solutions, we find the late-time slope of the pressure derivative as

$$v_{\text{no-flow}} = \lim_{t\to\infty}\frac{S_s}{4Kt}\frac{r^2\exp(-u) + r_i^2\exp(-u_i)}{\exp(-u) + \exp(-u_i)} \tag{17}$$

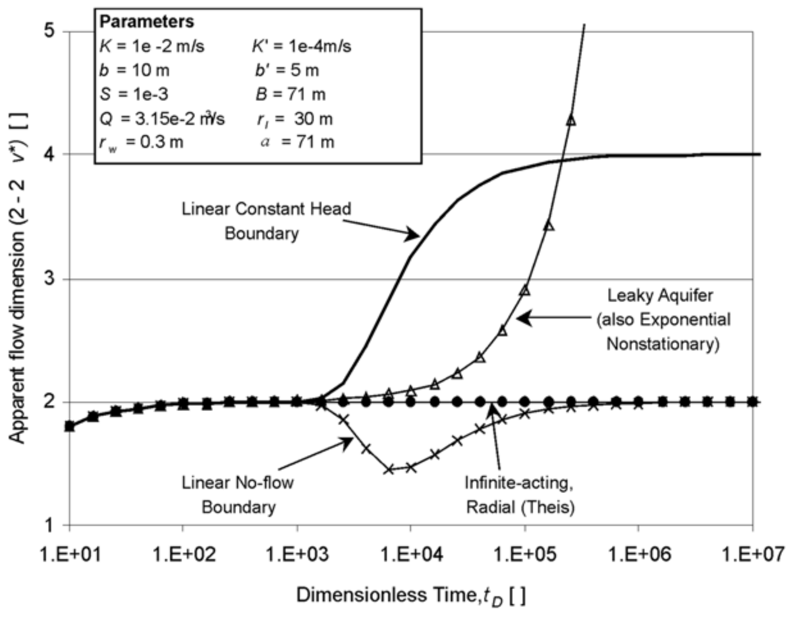

Figure 2. Exact solutions for the apparent flow dimension versus dimensionless time for drawdown at the pumped well $r_w = 0.3$ m ($r_D = 1$).

where the subscript i denotes terms corresponding to the image well, i.e.,

$$u_i = \frac{r_i^2 S_s}{4Kt} \tag{18}$$

and r_i is the radius from the image well to the observation point. Evaluating the limit of equation (17), $\nu_{\text{no-flow}} = 0$, and substituting into equation (6) yields the flow dimension $n = 2$, a result previously known via numerical differentiation [*Ehlig-Economides*, 1988]. Figure 2 shows the apparent flow dimension for a linear no-flow boundary 15 m from the pumping well (i.e., r_i = 30 m). The initial contact with the no-flow boundary causes the flow dimension to decrease, but the flow dimension eventually returns to n = 2. While this result may seem counterintuitive at first, recall that the flow dimension describes how the cross-sectional area of flow changes with radial distance. As the radius of influence of the test becomes large relative to the distance from the well to the no-flow boundary, the cross-sectional area of flow approaches $A(r) = \pi rb$, i.e., proportional to r^1, thus the flow dimension is n = 2. Similarly, a hydraulic test in a domain with a Euclidean dimension of two with intersecting no-flow boundaries (a wedge-shaped aquifer) will also yield a flow dimension of two at late time.

[14] Several authors [e.g., *Horne*, 1995] have noted that the pressure derivative for a constant-head boundary decreases continuously. For a linear constant-head boundary, we apply the same superposition strategy as was used to derive equation (17), and find that the late time slope of the pressure derivative is

$$\nu_{\text{const.head}} = \lim_{t\to\infty} \frac{S_s}{4Kt} \frac{r^2 \exp(-u) - r_i^2 \exp(-u_i)}{\exp(-u) - \exp(-u_i)}, \quad r \neq r_i \tag{19}$$

Evaluating the limit of equation (19) yields $\nu_{\text{const.head}} = -1$, and via equation (6) the flow dimension is found as n = 4 for a linear, constant-head boundary imposed on radial flow. Figure 2 shows (for a linear constant-head boundary 15 m from the pumping well) that the apparent flow dimension is n = 2 initially but increases under the influence of the constant-head boundary to asymptotically approach n = 4.

[15] The same strategy used to derive the flow dimension for the Theis problem (equation (7)) may be used to derive the flow dimension for radial flow in an infinite, homogeneous system under leaky conditions, i.e., the Hantush-Jacob leaky aquifer problem. The analytical solution for $h(r, t)$ in this case is presented by *Hantush and Jacob* [1955] as

$$h(r,t) = \frac{Q_0}{4\pi Kb} W(u, r/B) \tag{20}$$

where $B = (Kbb'/K')^{1/2}$ is the leakage factor [L], K' is the conductivity of the confining unit, b' is the difference between the overlying water table and the average head within the aquifer, and

$$W(u, r/B) = \int_u^\infty \frac{1}{y} \exp\left(-y - \frac{r^2}{4B^2 y}\right) dy. \tag{21}$$

$W(u, r/B)$ is also known as the leaky well function. Applying the same strategy as was used to derive equation (16), we find the late-time slope of the pressure derivative as

$$\nu_{\text{Hantush}} = \lim_{t\to\infty} \left[u - \frac{Kt}{B^2 S_s}\right] \tag{22}$$

Evaluating the limit of equation (22) yields $\nu_{\text{Hantush}} = -\infty$, and by equation (6) the corresponding flow dimension is $n = 2 - 2\nu_{\text{Hantush}}$, an unbounded increasing function at late time. For an impermeable confining layer, $K' \to 0$ so that $B \to \infty$ and equation (22) reduces equation (16), the slope of the pressure derivative for the Theis problem. This means that the apparent flow dimension can stabilize at n = 2 at early time and persists for an interval that increases with decreasing leakage. That is, slightly leaky systems at early times resemble nonleaky systems (Figure 2). At late time, the system is dominated by leakage and the flow dimension increases without bound. This is a consequence of the assumption of the Hantush problem that the tested aquifer is overlain by an unconfined aquifer of constant head (an essentially infinite source of water).

4. Solutions for Heterogeneous Transmissivity

[16] Hydraulic tests in heterogeneous aquifers are commonly interpreted by fitting the observed response with integer-valued flow models (n = 1, 2, or 3) and homogeneous parameters. The parameter estimates of these tests are acknowledged to be spatial averages whose averaging scale depends on the test duration and the hydraulic properties of the formation. A number of authors have examined the spatial averaging of hydraulic tests and the relationship between the interpreted values and the moments of the heterogeneous formation [e.g., *Butler*, 1991; *Meier et al.*, 1998; *Sánchez-Vila et al.*, 1999]. In contrast, the flow dimension of a heterogeneous medium has received comparatively little attention. *Chang and Yortsos* [1990] examined the fractal dimension of a fracture network, and *Acuna and Yortsos* [1995] showed that the fractal dimension of the network is a function of the flow dimension and a connectivity parameter. *Doughty and Karasaki* [2002] showed that the fractal dimension of a random Sierpinski lattice generally is greater than the flow dimension of a hydraulic test conducted in the lattice. The following section explores the flow dimensions of fields with specific forms of heterogeneous transmissivity. These include simple forms of stationary and nonstationary transmissivity, for which analytical and semianalytical solutions are readily obtained.

4.1. Nonstationary Case

[17] We begin by examining an aquifer with a linear trend in mean transmissivity arising from a thickness changing as $b(x) = b_0 + mx$, where b_0 is the thickness at the well at $x = 0$ (and homogeneous hydraulic conductivity). With a linear change m in thickness, the cross-sectional area of flow at a given radius is the surface area of a cylinder with a sloping bottom. This surface area can be determined by integrating along a parametric surface (adapted from *Grossman* [1992]):

$$A(r) = \int_s b(x) ds \tag{23}$$

where

$$ds = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta \tag{24}$$

Noting that the parametric curve describing a circle is $x = r\cos\theta$ and $y = r\sin\theta$, we can differentiate and substitute into equation (23) to get

$$A(r) = \int_0^{2\pi} [m(r\cos\theta) + b_0]\left[(-r\sin\theta)^2 + (-r\cos\theta)^2\right]^{1/2} d\theta. \tag{25}$$

Equation (25) solves to $A(r) = 2\pi r b_0$, the same cross-sectional area of flow as a standard radial system. This means that radial flow in a nonstationary field whose mean transmissivity has a simple linear trend will have a flow dimension of $n = 2$, the same as the homogeneous case. This is because the increasing thickness in the positive x-direction is balanced by an equivalent decrease in thickness in the negative x-direction, so the cross-sectional area of flow is still a first-power function of the radius, just as it is in a standard radial system.

[18] Of course, the form of the nonstationarity need not be linear; consider an aquifer whose transmissivity decreases due to an exponential decrease in thickness with distance in the positive x-direction. Figure 3 is a schematic of this form of nonstationarity, whose thickness varies as

$$b(x) = b_0 \exp\left(\frac{-2(x - x_0)}{a}\right) \tag{26}$$

where b_0 is the aquifer thickness at the well [L], x_0 is the well location along x [L], and a is a parameter that controls the rate of change in thickness [L]. For this situation, *Hantush* [1962] gave the solution for $h(r, t)$ as

$$h(r,t) = \frac{Q_0}{4\pi K b_0} \exp\left(\frac{r}{a}\cos\theta\right) W(u, r/a) \tag{27}$$

where $W(u, r/a)$ is the leaky well function (equation (20)) with the parameter a substituted in for B, the leakage factor (valid for $db/dx < 0.20$). Applying again the strategy used to derive equations (16) and (22), the exact late-time slope of the pressure derivative is

$$v_{\mathrm{nonstationary}} = \lim_{t\to\infty}\left[u - \frac{Kt}{a^2 S_s}\right] \tag{28}$$

Evaluating the limit of equation (28) yields $v_{\mathrm{nonstationary}} = -\infty$, and thus the flow dimension $n = 2 - 2v_{\mathrm{nonstationary}}$ is an unbounded, increasing function of time. That is, the flow dimension for an exponential form of nonstationary transmissivity is greater than two and increases without bound. Comparing equations (22) and (28) reveals that this particular form of nonstationary transmissivity yields a flow dimension that is identical to that of a homogeneous system under leaky conditions with $B = a$ (Figure 2).

4.2. Stationary Case

[19] In this section we determine the flow dimension of an idealized model of stationary heterogeneity, that of a periodic transmissivity with a modest level of variability. The starting point for this analysis is the semianalytical solution of *Oliver* [1990] for the pressure derivative for a constant-rate hydraulic test in an infinite heterogeneous medium. *Oliver* [1990] developed this approximate solution using a first-order perturbation in permeability. In terms of hydraulic conductivity, Oliver's solution is

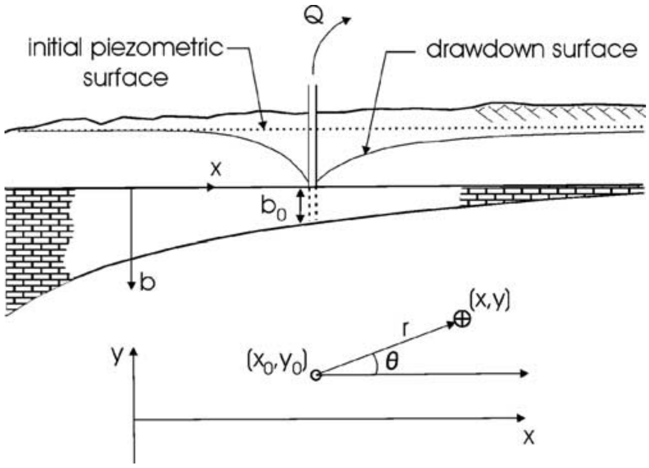

Figure 3. Schematic of the hydrogeologic system addressed by the *Hantush* [1962] solution for a nonstationary transmissivity of exponential form.

$$\frac{\partial p_D}{\partial \ln t_D} = -\frac{1}{2} - \frac{1}{2\pi}\int_1^{\infty} \Phi(\xi, t_D) \int_{-\pi}^{\pi} \varepsilon \cdot f(\xi,\theta)\, d\theta\, d\xi \tag{29}$$

$$\varepsilon = \frac{\bar{K}}{K(r_D,\theta)} - 1, \quad f(\xi,\theta) = \begin{cases} 1 & (\xi,\theta) = (r_{D\varepsilon}, \theta_\varepsilon) \\ 0 & \text{otherwise} \end{cases}$$

where $p_D(r_D, t_D) = h(r_D, t_D) \cdot 2\pi\bar{K}b/Q_0$ is the dimensionless pressure change, and $r_D = r/r_w$ is the dimensionless radius. The filter function is defined as

$$\Phi(\xi, t_D) = \frac{\xi\sqrt{\pi}}{2t_D}\exp\left[-\frac{\xi^2}{2t_D}\right] \cdot \mathrm{W}_{1/2,1/2}\left(\frac{\xi^2}{t_D}\right). \tag{30}$$

The location function $f(\xi, \theta)$ represents the spatial distribution of ε, the perturbation of hydraulic conductivity relative to $\bar{K}$, the mean hydraulic conductivity of the field. The Whitaker function, $\mathrm{W}_{1/2,1/2}(z)$, is the confluent hypergeometric function [*Whitaker and Watson*, 1927]. The averaging effect of the hydraulic test is incorporated by the filter function, $\Phi(\xi, t_D)$, which expands with time, weighting the values of the heterogeneous field as the test progresses. The Whitaker function $\mathrm{W}_{1/2,1/2}(z)$ is approximated via the first few terms of the infinite series given by *Oliver* [1990], which converges for large dimensionless times.

[20] Oliver's solution (equation (29)) is particularly useful because the expression $\varepsilon \cdot f(\xi, \theta)$ represents an arbitrary spatial distribution of hydraulic conductivity contrasts. Equation (29) thus provides an approximate solution for $dh/d\ln t$ for a constant-rate, infinite system of arbitrarily

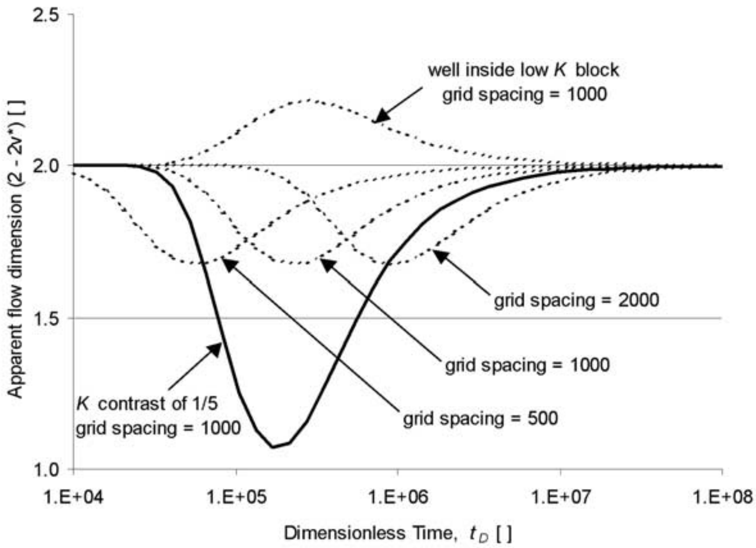

Figure 4. Approximate solutions for the apparent flow dimension versus dimensionless time for the stationary, heterogeneous case, for drawdown at the pumped well ($r_D = 1$). Dashed lines correspond to a 1/2 decrease in K; grid spacings are given in dimensionless length $\Delta x_D = \Delta x / r_w$.

heterogeneous transmissivity. However, the first-order perturbation approach used to develop this solution limits its applicability to small contrasts in hydraulic conductivity. Oliver applied equation (29) to a radial composite system, where the hydraulic conductivity changes as a step function of the radius (i.e., a hydraulic conductivity contrast occurring as a ring around the well). Oliver found that equation (29) is approximately correct for at least a factor of 1/5 perturbation of hydraulic conductivity and thus should be sufficiently accurate for the purposes of this paper.

[21] Estimating the flow dimension of a heterogeneous medium is accomplished by relatively straightforward numerical analysis of equation (29). The nested integrals of equation (29) are approximately solved using an adaptive Gauss-Kronrod numerical quadrature [*Press et al.*, 1992], yielding the dimensionless pressure derivative at a particular dimensionless time. The solution is repeated for a sequence of dimensionless times, and the late-time slope of the pressure derivative is determined by numerical differentiation (central finite difference) of the common logarithm of the sequence. The flow dimension is computed via equation (6), as discussed above.

[22] The above numerical approach for estimating the flow dimension is applied to an idealized case of stationary heterogeneity: an infinite, periodic transmissivity field of modest variability. For this example, the thickness b is uniform and hydraulic conductivities alternate between $\bar{K}$ and $0.5\bar{K}$ on a regular grid pattern (i.e., a checkerboard pattern in K with a 1:2 contrast ratio), with the well centered in a block of $K = \bar{K}$ and a grid spacing of $\Delta x_D = 1000$. The flow dimension is calculated at the pumped well, radius $r_D = 1.0$. Figure 4 shows that as the pressure diffuses, the transmissivity changes are eventually averaged over a sufficiently large enough area that the flow dimension converges to $n = 2$. The calculation is repeated for grid spacings of $\Delta x_D = 500$ and 2000, whose results indicate that the convergence is delayed and is slower as the characteristic scale of the heterogeneity increases. The calculation is repeated for a contrast of 1:5 and $\Delta x_D = 1000$, whose results show that the deviation in the flow dimension increases with increasing contrast (Figure 4). The calculation is repeated for $\Delta x_D = 1000$ and phase shift of 1/2 of the medium's period (i.e., the well is centered in a block of decreased conductivity $K = 0.5\bar{K}$), resulting in a corresponding reversal of the deviation of the flow dimension.

5. Summary and Discussion

[23] In this paper we have summarized the GRF approach to hydraulic test interpretation and have shown how the flow dimension n is used to adapt the GRF interpretive model to hydrogeologic conditions. We have discussed how system geometry and heterogeneities can influence the flow dimension of a system and have reviewed the mathematical basis for the interchangeability of geometry and heterogeneity as influences on the flow dimension interpreted from a well test. Since the flow dimension is not necessarily the same as the Euclidean dimension of the tested medium, we have pursued the objective of establishing the flow dimensions for common hydrogeologic conditions. All the cases examined in this paper are for constant-rate hydraulic tests in domains with a Euclidean dimension of two.

[24] We have examined analytical solutions for hydraulic tests within the context of the log-log diagnostic plot of *Bourdet et al.* [1983a] to determine the flow dimensions for several hydrogeologic conditions. We verified the flow dimensions for the known cases of infinite-acting radial flow and radial flow with a linear no-flow boundary, and then found the exact solutions for the flow dimensions of several cases of interest. Imposing a linear, constant-head boundary on infinite-acting radial flow eventually yields a flow dimension of four, and the Hantush solution for a leaky aquifer yields a flow dimension that is a function of time

and the leakage factor. We have shown that the flow dimension of nonstationary fields depends on the form and magnitude of the nonstationarity. We have also shown that a nonstationary transmissivity of exponential form can yield a flow dimension that is indistinguishable from that of a leaky aquifer.

[25] We applied the approximate, semianalytical solution of *Oliver* [1990] for a constant-rate hydraulic test in a heterogeneous field to a periodic transmissivity as an idealized case of stationary heterogeneity. This case showed that the flow dimension of a stationary transmissivity field stabilizes at $n = 2$ and that the stabilization rate depends on the magnitude and the length scale (period) of the heterogeneity. These results suggest that for a stationary heterogeneous transmissivity field of modest variability, the underlying dimensional assumptions of a radial interpretive model are reassuringly consistent with the observed flow dimension (given sufficient test duration).

[26] We have found that the flow dimension of a hydraulic test can reflect several hydrogeologic conditions and that some of these conditions can yield identical flow dimensions. This is in agreement with *Doe* [1991], who pointed out that heterogeneity and flow geometry are interchangeable, and this implies that the apparent flow dimension cannot be uniquely interpreted. This nonuniqueness might be viewed as a limitation of the GRF approach, but it also permits an examination of alternative conceptual models. For example, this paper has shown that a leaky aquifer and an aquifer with an exponential nonstationary transmissivity can yield the same apparent flow dimensions. There are a number of hydrogeologic settings where both conditions could be expected to exist, such as the aquifers within the glacial deposits of the northern midwestern United States. In this setting, the thickness of productive gravel aquifers can vary dramatically over short distances and also can be influenced by leakage. The apparent flow dimension of a well test in such a system might be interpreted reasonably as the result of either leakage, or nonstationary transmissivity, or some combination of both. In the absence of additional information that might eliminate one of these interpretations, subsequent analyses of such systems must address both conceptual models independently as well as their plausible combinations. That is, even though the effects of multiple hydrogeologic conditions can confound test interpretation, the flow dimension constrains the admissible combinations of conceptual models for the tested system. The GRF approach thus requires placing hydraulic testing in the full context of the hydrogeologic setting, rather than forcing the test into the assumptions of the interpretive model. The observed flow dimension needs to be combined with all other knowledge of the system (geologic, geophysical, hydrologic, etc.) to construct a meaningful conceptual model of the system. Similarly, the calibration of numerical models of these systems should confirm that the numerical model reproduces the flow dimension observed from field hydraulic tests.

[27] **Acknowledgments.** The authors are grateful to James Butler for his constructive criticism of the preliminary draft of this paper. The authors are also indebted to the two anonymous reviewers of this article, whose insightful comments led to substantial improvements in the paper. Sandia is a multiprogram laboratory operated by Sandia Corporation, a Lockheed Martin Company, for the United States Department of Energy under contract DE-AC04-94AL85000.

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